5.3: The Definite Integral

Def^n: The integral of f over $[a,b]$ is given by

$$\int_a^b f(x)\,dx = \text{limit of Riemann Sums}$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a+k\cdot\tfrac{b-a}{n}\right)\cdot \tfrac{b-a}{n}. \text{ (if it exists).}$$

This is called the <u>definite integral</u>.

If the limit exists, we say f is <u>integrable</u> (on $[a,b]$).

<u>Theorem</u>: If f is continuous (or has at most finitely many jump discontinuities) on $[a,b]$, then f is integrable.

<u>Example of non-integrable fcn</u>:

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

<u>Theorem 2: (Rules of Integration)</u>

(i) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(ii) $\int_a^a f(x)\,dx = 0$

(iii) $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

(iv) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(v) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(vi) $\min(f)\cdot(b-a) \le \int_a^b f(x)\,dx \le \max f\cdot(b-a)$

(vii) If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Ex(2): Suppose $\int_{-1}^{1} f(x)\,dx = 5$, $\int_{1}^{4} f(x)\,dx = -2$, $\int_{-1}^{1} h(x)\,dx = 7$

(a) $\int_{4}^{1} f(x)\,dx = 2$

(b) $\int_{-1}^{1} 2f(x) + 3h(x)\,dx = 2\cdot 5 + 3\cdot 7 = 31$

(c) $\int_{-1}^{4} f(x)\,dx = 5 + (-2) = 3.$

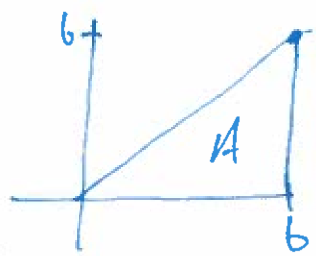
Ex(3): Show that $\int_{0}^{1} \sqrt{1+\cos x}\,dx \leq \sqrt{2}.$

$\text{Max}(\sqrt{1+\cos x}) \leq \sqrt{2}.$ So $\int_{0}^{1} \sqrt{1+\cos x}\,dx \leq \sqrt{2}\cdot(1-0) = \sqrt{2}.$

Remark: If f is integrable and non-negative on $[a,b]$ then the area under the curve, A, satisfies

$$A = \int_{a}^{b} f(x)\,dx.$$

Ex(4): Find $\int_{0}^{b} x\,dx.$



$A = \int_{0}^{b} x\,dx = \frac{1}{2} b \cdot b = \frac{b^2}{2}.$

Def^n: The average value of f over $[a,b]$ is given by

$$\text{Average} = \frac{1}{(b-a)} \cdot \int_{a}^{b} f(x)\,dx.$$

This can be shown using Riemann Sums.

This is also called the mean of f over $[a,b]$.